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# Discontinuous implicit generalized quasi-variational inequalities in Banach spaces

**Paolo Cubiotti · Jen-chih Yao**

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**Abstract** We consider the following implicit quasi-variational inequality problem: given two topological vector spaces $E$ and $F$, two nonempty sets $X \subseteq E$ and $C \subseteq F$, two multifunctions $\Gamma\colon X \to 2^X$ and $\Phi\colon X \to 2^C$, and a single-valued map $\psi: X \times C \times X \to \mathbb{R}$, find a pair $(\hat{x}, \hat{z}) \in X \times C$ such that $\hat{x} \in \Gamma(\hat{x})$, $\hat{z} \in \Phi(\hat{x})$ and $\psi(\hat{x}, \hat{z}, y) \leq 0$ for all $y \in \Gamma(\hat{x})$. We prove an existence theorem in the setting of Banach spaces where no continuity or monotonicity assumption is required on the multifunction $\Phi$. Our result extends to non-compact and infinite-dimensional setting a previous results of the authors (Theorem 3.2 of Cubbiotti and Yao [15] Math. Methods Oper. Res. **46**, 213–228 (1997)). It also extends to the above problem a recent existence result established for the explicit case ($C = E^*$ and $\psi(x, z, y) = \langle z, x - y\rangle$).

**Keywords** Implicit generalized quasi-variational inequalities · Multifunctions · Lower semicontinuity · Upper semicontinuity · Banach space

## 1 Introduction

The importance of the variational inequality theory is well documented in the literature, due to its wide range of applications to fields of mathematics such as mechanics, network equilibrium, control theory, game theory, complementarity problems, optimization, etc. (see for instance [2–5, 19, 22–26, 30, 31, 33, 38, 39]).

Let $E$ and $F$ be two topological vector spaces, $X \subseteq E$ and $C \subseteq F$ two nonempty sets, $\Gamma\colon X \to 2^X$ and $\Phi\colon X \to 2^C$ two multifunctions, and $\psi\colon X \times C \times X \to \mathbb{R}$ a single-valued map. The *implicit generalized quasi-variational inequality problem* associated

P. Cubiotti
Department of Mathematics, University of Messina, Contrada Papardo, Salita Sperone 31, 98166 Messina, Italy

J.-c. Yao (✉)
Department of Applied Mathematics, National Sun Yat-sen University, Kaohsiung 80424, Taiwan R.O.C.
e-mail: yaojc@mail.math.nsysu.edu.tw

with $X$, $C$, $\Gamma$, $\Phi$ and $\psi$ (see [15, 27]) is to find a pair $(\hat{x}, \hat{z}) \in X \times C$ such that

$$\hat{x} \in \Gamma(\hat{x}), \quad \hat{z} \in \Phi(\hat{x}) \quad \text{and} \quad \psi(\hat{x}, \hat{z}, y) \leq 0 \quad \text{for all} \quad y \in \Gamma(\hat{x}). \tag{1}$$

When $C = F = E^*$ and $\psi(x,z,y) = \langle z, x-y \rangle$ ($E^*$ denoting the topological dual of $E$), problem (1) gives back the usual generalized quasi-variational inequality problem [6, 35], which, in turn, extends the classical variational inequality problem [26].

It is interesting to observe that problem (1) contains as a special case the extended generalized quasi-variational inequality problem introduced and studied by one of the authors in [39], motivated by several applications to game and economic theory. It also contains the variational-like inequality problem studied in [32, 33, 38].

As regards existence results for generalized quasi-variational inequality problems (both in explicit and implicit form), a very common assumption in the literature is the upper semicontinuity of the multifunction $\Phi$. When $\Phi$ is single-valued, such an assumption reduces to the ordinary notion of continuity. Recently, much attention has been paid by many authors to the case where the multifunction $\Phi$ has no continuity properties (see for instance [7, 8, 10, 11, 13, 14, 16, 17, 29, 34, 37, 40–42]), obtaining new existence results and also applications to fixed point theory, control theory and to the traffic equilibrium problem [9, 12, 18, 20].

In the paper [15], problem (1) has been studied in the case where $E$ and $F$ are finite-dimensional, obtaining some existence results without continuity assumptions on the multifunction $\Phi$ and also applications to generalized quasi-variational inequalities with discontinuous fuzzy mappings.

Our aim in this paper is to extend the main result of Cubiotti and Yao [15] (Theorem 3.2) to noncompact and infinite-dimensional setting. We prove an existence theorem in the setting of Banach spaces where no continuity assumption is required on the multifunction $\Phi$ (Theorem 3.3 below). It should be mentioned that a first attempt in the same direction has been made recently in the paper [27]. Even though our result in this paper improves some aspects of the main result of Huang and Yao [27], it can be checked that the two results are independent.

Finally, we remark that our result covers as a special case (and also improves in some aspects) a recent existence result established for the explicit case (Theorem 3.1. of Cubiotti [14]).

## 2 Preliminaries

For the basic facts about multifunctions, we refer to Klein and Thompson [28]. Here, we only recall that given two topological spaces $S$ and $Y$ and a multifunction $F\colon S \to 2^Y$, we say that $F$ is lower semicontinuous (resp., upper semicontinuous) at $x \in S$ if for each open set $A \subseteq Y$, with $F(x) \cap A \neq \emptyset$ (resp., with $F(x) \subseteq A$), the set $F^-(A) := \{s \in S : F(s) \cap A \neq \emptyset\}$ (resp., the set $\{s \in S : F(s) \subseteq A\}$) is a neighborhood of $x$ in $S$. We say that $F$ is lower (resp., upper) semicontinuous in $S$ if it is lower (resp., upper) semicontinuous at each point $x \in S$. The graph of $F$ is the set $\{(s,y) \in S \times Y : y \in F(s)\}$.

Let $(E, \|\cdot\|_E)$ be a real normed space. If $x \in E$ and $r > 0$, we denote by $B(x,r)$ and $\overline{B}(x,r)$, respectively, the open ball and the closed ball in $E$ centered at $x$ with radius $r$. For simplicity, we shall put

$$B_r := B(0,r), \qquad \overline{B}_r := \overline{B}(0,r).$$

We say that a multifunction $F\colon\ S \to 2^E$ is Hausdorff lower semicontinuous (resp., Hausdorff upper semicontinuous) at $x_0 \in S$ if for each $\epsilon > 0$ there exists a neighborhood $U$ of $x_0$ in $S$ such that

$$F(x_0) \subseteq F(x) + B_\epsilon \quad \text{for all} \quad x \in U,$$
$$\big(\text{resp.,} \quad F(x) \subseteq F(x_0) + B_\epsilon \quad \text{for all} \quad x \in U\,\big).$$

We say that $F$ is Hausdorff lower (resp., Hausdorff upper) semicontinuous in $S$ if it is Hausdorff lower (resp., Hausdorff upper) semicontinuous at each point $x \in S$. We say that $F$ is Hausdorff continuous if it is Hausdorff lower and upper semicontinuous. It is easy to check [28, 36] that Hausdorff lower semicontinuity implies lower semicontinuity, and, conversely, upper semicontinuity implies Hausdorff upper semicontinuity. The converse implications are true if each set $F(x)$ is nonempty and compact [28, Theorem 7.1.14].

If $A \subseteq E$ is a nonempty set and $x \in E$, we put

$$d(x, A) := \inf_{u \in A} \|x - u\|_E.$$

Moreover, we shall denote by $\mathrm{aff}(A)$ the affine hull of the set $A$. If $A \subseteq C \subseteq E$, we shall denote by $\mathrm{int}_C(A)$ the interior of $A$ in $C$. We recall that if $A \subseteq E$ is a nonempty finite-dimensional convex set, then $\mathrm{int}_{\mathrm{aff}(A)}(A) \neq \emptyset$. Finally, we recall that the set $A \subseteq E$ is said to be compactly closed (resp., finitely closed) if its intersection with any compact subset (resp., any finite-dimensional subspace) of $E$ is closed.

## 3 Results

Before stating our main existence theorem, we give some preliminary results. First, we note that Theorem 3.2 of Cubiotti and Yao [15] still holds if the set $C \subseteq \mathbb{R}^m$ is replaced by any nonempty subset of an Hausdorff topological vector space $F$. Starting from this, we can obtain the following noncompact version of the same result (where the open and closed balls are taken obviously in the space $\mathbb{R}^n$ and $\|\cdot\|$ denotes the Euclidean norm of $\mathbb{R}^n$).

**Theorem 3.1** *Let $X \subseteq \mathbb{R}^n$ be a nonempty closed convex set, $C$ a nonempty subset of the Hausdorff topological vector space $F$, $\Gamma\colon\ X \to 2^X$ and $\Phi\colon\ X \to 2^C$ two multifunctions, $\psi\colon\ X \times C \times X \to \mathbb{R}$ a single-valued map. Assume that as follows:*

(1) $\Gamma$ *is lower semicontinuous with nonempty convex values;*
(2) *the set $E := \{x \in X : x \in \Gamma(x)\}$ is closed;*
(3) $\mathrm{aff}(\Gamma(x)) = \mathrm{aff}(X)$ *for all $x \in E$;*
(4) $\Phi(x)$ *is nonempty and compact for $x \in X$ and convex for $x \in E$;*
(5) *for each $y \in X$, the set $\{x \in E : \inf_{z \in \Phi(x)} \psi(x, z, y) \leq 0\}$ is closed;*
(6) *for each $x \in E$, the set $\{y \in X : \inf_{z \in \Phi(x)} \psi(x, z, y) \leq 0\}$ is closed;*
(7) *for each $x \in E$ and each $z \in \Phi(x)$, one has $\psi(x, z, x) = 0$;*
(8) *for each $x \in E$ and each $z \in \Phi(x)$, the function $\psi(x, z, \cdot)$ is concave on $\Gamma(x)$;*
(9) *for each $x \in E$ and each $y \in \Gamma(x)$, the function $\psi(x, \cdot, y)$ is lower semicontinuous (in the sense of single-valued maps) and convex on $\Phi(x)$.*

*Moreover, assume that there exists $r > 0$ such that the following conditions hold:*

(10) *$X \cap B_r \neq \emptyset$ and $\Gamma(x) \cap B_r \neq \emptyset$ for all $x \in X \cap \overline{B}_r$;*
(11) *for each $x \in E$, with $\|x\| = r$, and each $z \in \Phi(x)$, there exists $y \in \Gamma(x)$, with $\|y\| < r$, such that $\psi(x, z, y) \geq 0$.*

*Then there exists $(\hat{x}, \hat{z}) \in X \times C$, with $\hat{x} \in \Gamma(\hat{x})$, $\hat{z} \in \Phi(\hat{x})$ and $\|\hat{x}\| \leq r$, such that*

$$\psi(\hat{x}, \hat{z}, y) \leq 0 \quad \text{for all} \quad y \in \Gamma(\hat{x}).$$

*Proof* Put $X_r := X \cap \overline{B}_r$. By Proposition 2.1 of Cubiotti and Yuan [17], the multifunction

$$\Gamma_r\colon\ x \in X_r \to \Gamma(x) \cap X_r = \Gamma(x) \cap \overline{B}_r$$

is lower semicontinuous with nonempty convex values. Moreover, its fixed-points set

$$E_r := \{x \in X_r :\ x \in \Gamma(x) \cap \overline{B}_r\} = E \cap \overline{B}_r$$

is closed by (2). We claim that

$$\operatorname{aff}(\Gamma_r(x)) = \operatorname{aff}(X_r) \quad \text{for all} \quad x \in E_r.$$

To prove this, fix any $x \in E_r$. Since, by convexity one has $\overline{\Gamma(x)} = \overline{\operatorname{ri}(\Gamma(x))}$, by assumption (10), we get

$$\operatorname{ri}(\Gamma(x)) \cap B_r \neq \emptyset.$$

Choose any point $u \in \operatorname{ri}(\Gamma(x)) \cap B_r$, and let $\epsilon > 0$ be such that

$$B(u, \epsilon) \subseteq B_r \quad \text{and} \quad B(u, \epsilon) \cap \operatorname{aff}(\Gamma(x)) \subseteq \Gamma(x).$$

Since, $B(u, \epsilon) \cap \operatorname{aff}(\Gamma(x))$ is open in $\operatorname{aff}(\Gamma(x))$, its affine hull coincides with the whole $\operatorname{aff}(\Gamma(x))$, that is

$$\operatorname{aff}(B(u, \epsilon) \cap \operatorname{aff}(\Gamma(x))) = \operatorname{aff}(\Gamma(x)).$$

Moreover, since

$$B(u, \epsilon) \cap \operatorname{aff}(\Gamma(x)) \subseteq \Gamma(x) \cap B_r$$

taking into account (3), we get

$$\begin{aligned}\operatorname{aff}(B(u, \epsilon) \cap \operatorname{aff}(\Gamma(x))) &\subseteq \operatorname{aff}(\Gamma(x) \cap B_r) \subseteq \operatorname{aff}(\Gamma(x) \cap \overline{B}_r) \subseteq \operatorname{aff}(X \cap \overline{B}_r)\\ &\subseteq \operatorname{aff}(X) = \operatorname{aff}(\Gamma(x)) = \operatorname{aff}(B(u, \epsilon) \cap \operatorname{aff}(\Gamma(x))).\end{aligned}$$

In particular, this implies that

$$\operatorname{aff}(\Gamma(x) \cap \overline{B}_r) = \operatorname{aff}(X \cap \overline{B}_r),$$

as claimed. Consequently, applying Theorem 3.2 of Cubiotti and Yao [15] to the set $X_r$ and to the multi-functions $\Gamma_r$ and $\Phi|_{X_r}$ (it is routine matter to check that all the remaining assumptions are satisfied), we get the existence of a pair $(\hat{x}, \hat{z}) \in (X \cap \overline{B}_r) \times C$ such that

$$\hat{x} \in \Gamma(\hat{x}), \quad \hat{z} \in \Phi(\hat{x}) \quad \text{and} \quad \psi(\hat{x}, \hat{z}, v) \leq 0 \quad \text{for all} \quad v \in \Gamma(\hat{x}) \cap \overline{B}_r. \tag{2}$$

We now show that the pair $(\hat{x}, \hat{z})$ satisfies the conclusion. To see this, fix any $y \in \Gamma(\hat{x})$. We distinguish two cases.

(a) $\|\hat{x}\| = r$. By assumption (xi), there exists a point $w \in \Gamma(\hat{x}) \cap B_r$ such that $\psi(\hat{x},\hat{z},w) \geq 0$. By (2), we get $\psi(\hat{x},\hat{z},w) = 0$. Choose $t \in ]0,1[$ in such a way that

$$y_t := t\,y + (1-t)w \in \overline{B}_r.$$

Taking into account (2), assumption (8) and the convexity of $\Gamma(\hat{x})$, we get

$$0 \geq \psi(\hat{x},\hat{z},y_t) \geq t\,\psi(\hat{x},\hat{z},y) + (1-t)\,\psi(\hat{x},\hat{z},w) = t\,\psi(\hat{x},\hat{z},y)$$

hence

$$\psi(\hat{x},\hat{z},y) \leq 0$$

as claimed.

(b) $\|\hat{x}\| < r$. Choose $t \in ]0,1[$ in such a way that

$$u_t := ty + (1-t)\hat{x} \in \overline{B}_r.$$

Taking into account (2), assumptions (7) and (8), and the convexity of $\Gamma(\hat{x})$, we get

$$0 \geq \psi(\hat{x},\hat{z},u_t) \geq t\,\psi(\hat{x},\hat{z},y) + (1-t)\,\psi(\hat{x},\hat{z},\hat{x}) = t\,\psi(\hat{x},\hat{z},y)$$

hence

$$\psi(\hat{x},\hat{z},y) \leq 0,$$

as claimed. □

**Corollary 3.2** *Let $X, C, F, \Gamma, \Phi, \psi$ be as in Theorem* 3.1. *Assume that assumptions* (1)–(9) *of Theorem* 3.1 *are satisfied. Moreover, assume that there exists a nonempty compact set $K \subseteq X$ such that the following conditions hold:*

(10)′ $\Gamma(x) \cap K \neq$ *for all $x \in X$;*
(11)′ *for each $x \in E \backslash K$, and each $z \in \Phi(x)$, there exists $y \in \Gamma(x) \cap K$ such that $\psi(x,z,y) \geq 0$.*

*Then there exists $(\hat{x},\hat{z}) \in X \times C$, with $\hat{x} \in \Gamma(\hat{x})$ and $\hat{z} \in \Phi(\hat{x})$, such that*

$$\psi(\hat{x},\hat{z},y) \leq 0 \quad \textit{for all} \quad y \in \Gamma(\hat{x}).$$

*Proof* It suffices to take $r > 0$ in such a way that $K \subseteq B_r$. The conclusion follows at once by Theorem 3.1. □

The following is the main result of the paper.

**Theorem 3.3** *Let $(E, \|\cdot\|_E)$ be a real Banach space, $X \subseteq E$ a closed convex set, $C$ a nonempty subset of the Hausdorff topological vector space $F$, $\Gamma\colon X \to 2^X$ and $\Phi\colon X \to 2^C$ two multifunctions, $\psi\colon X \times C \times X \to \mathbb{R}$ a single-valued map. Let $K_1, K_2 \subseteq X$ be two nonempty compact sets, such that $K_1 \subseteq K_2$ and $K_1$ is finite-dimensional. Assume that:*

(1) *the multifunction $\Gamma$ is Hausdorff lower semicontinuous with closed convex values;*
(2) *the set $E := \{x \in X : x \in \Gamma(x)\}$ is compactly closed;*
(3) $\mathrm{int}_{\mathrm{aff}(X)}(\Gamma(x)) \neq \emptyset$ *for all $x \in X$;*
(4) $\Phi(x)$ *is nonempty and compact for $x \in X$ and convex for $x \in E$;*
(5) *for each $y \in X$, the set $\{x \in E : \inf_{z\in\Phi(x)} \psi(x,z,y) \leq 0\}$ is compactly closed;*
(6) *for each $x \in E$, the set $\{y \in X : \inf_{z\in\Phi(x)} \psi(x,z,y) \leq 0\}$ is finitely closed;*

(7) *for each* $x \in E$ *and each* $z \in \Phi(x)$, *one has* $\psi(x,z,x) = 0$;
(8) *for each* $x \in E$ *and each* $z \in \Phi(x)$, *the function* $\psi(x,z,\cdot)$ *is concave on* $\Gamma(x)$ *and the set* $\{y \in X : \psi(x,z,y) \le 0\}$ *is closed;*
(9) *for each* $x \in E$ *and each* $y \in \Gamma(x)$, *the function* $\psi(x,\cdot,y)$ *is lower semicontinuous* (*in the sense of single-valued maps*) *and convex on* $\Phi(x)$.
(10) $\Gamma(x) \cap K_1 \neq \emptyset$ *for all* $x \in X$;
(11) *for each* $x \in E \setminus K_2$, *and each* $z \in \Phi(x)$, *one has*

$$\sup_{y \in \Gamma(x) \cap K_1} \psi(x,z,y) > 0.$$

*Then there exists* $(\hat{x},\hat{z}) \in K_2 \times C$, *with* $\hat{x} \in \Gamma(\hat{x})$ *and* $\hat{z} \in \Phi(\hat{x})$, *such that*

$$\psi(\hat{x},\hat{z},y) \le 0 \quad \textit{for all} \quad y \in \Gamma(\hat{x}).$$

**Remark** When $C = F = E^*$ and $\psi(x,z,y) = \langle z, x-y \rangle$, the assumption (6) of Theorem 3.3 is automatically satisfied. Indeed, for each fixed $x \in E$ the function

$$y \to \inf_{z \in \Phi(x)} \langle z, x-y \rangle$$

being concave, is continuous over any finite-dimensional subspace of $E$. Moreover, we note that if the multifunction $\Gamma$ has closed graph, then it has closed values and the set $E$ is closed, while the converse implication is not true in general. Consequently, taking $C = F = E^*$ (endowed with the weak-star topology) and $\psi(x,z,y) = \langle z, x-y \rangle$, Theorem 3.3 gives back (and also improves) Theorem 1.2 of Cubiotti [14]. Finally, we remark that the completeness of the space $E$ is necessary only to ensure that $\overline{\text{co}}\, K_2$ (the closed convex hull of the set $K_2$) is compact. Therefore, Theorem 3.3 still holds if $E$ is any real normed space and $\overline{\text{co}}\, K_2$ is compact.

*Proof of Theorem* 3.3 Let $V := \text{aff}(X)$ (the affine hull of $X$), and let $V_0$ be the linear subspace of $E$ corresponding to $V$ (of course, $V$ may not be closed in $E$). For each $z \in \overline{\text{co}}\, K_2$, choose any point $u_z \in \text{int}_V\, \Gamma(z)$ (the interior of $\Gamma(z)$ in $V$), which is nonempty by assumption (3). By Proposition 2.5 of Cubiotti [13], for each $z \in \overline{\text{co}}\, K_2$ there exists an open bounded neighborhood $W_z$ of $z$ in $E$ such that

$$u_z \in \text{int}_V \left( \bigcap_{v \in W_z \cap X} \Gamma(v) \right). \tag{3}$$

By Theorem 6 at p416 of Dunford and Schwartz [21], the set $\overline{\text{co}}\, K_2$ is compact. Therefore, we can find points $z_1, \ldots, z_m \in \overline{\text{co}}\, K_2$ such that

$$\overline{\text{co}}\, K_2 \subseteq \Sigma_1 := \bigcup_{i=1}^{m} \left[ W_{z_i} \cap V \right]. \tag{4}$$

First, we note that $\Sigma_1$ is open in $V$ and bounded. Therefore, since $V \setminus \Sigma_1 \neq \emptyset$, the set $V \setminus \Sigma_1$ is closed in $V$ and $\overline{\text{co}}\, K_2$ is compact, by (4), we get

$$\xi := \inf \left\{ d(a, V \setminus \mathcal{S}_1) : a \in \overline{\text{co}}\, K_2 \right\} > 0. \tag{5}$$

If we put

$$\Sigma := \overline{\text{co}}\, K_2 + \left[ \overline{B}_{\frac{\xi}{2}} \cap V_0 \right], \tag{6}$$

we have that $\Sigma$ is convex and closed in $V$, and also $\Sigma \subseteq \Sigma_1$.

Let $\mathcal{S}$ be the family of all finite-dimensional linear subspaces of $E$ containing the set

$$K_1 \cup \{u_{z_1}, \ldots, u_{z_m}\}.$$

Fix $S \in \mathcal{S}$, and put

$$X_S := \overline{X \cap \Sigma \cap S}.$$

Observe that

$$K_1 \subseteq X \cap \Sigma \cap S \subseteq X_S \subseteq X \cap S.$$

In particular, $X_S \neq \emptyset$. Let $\Gamma_S$: $X_S \to 2^{X_S}$ be the multifunction defined by setting, for each $x \in X_S$,

$$\Gamma_S(x) := \Gamma(x) \cap X_S = \Gamma(x) \cap \overline{X \cap \Sigma \cap S}.$$

At this point, our aim is to apply Corollary 3.2 to the data $X_S, C, \Gamma_S, \Phi|_{X_S}, \psi|_{X_S \times C \times X_S}$. To this aim, we observe the following facts.

(a) The set $X_S$ is a nonempty closed convex subset of $S$.
(b) The multifunction $\Gamma_S$: $X_S \to 2^{X_S}$ has nonempty convex values by (1) and (10) (since $K_1 \subseteq X_S$). Moreover, since one has

$$E_S := \left\{x \in X_S : \ x \in \Gamma_S(x)\right\} = E \cap X_S$$

and the set $E$ is finitely closed by (2), it follows that the set $E_S$ is closed.
(c) The multifunction $\Gamma_S$: $X_S \to 2^{X_S}$ is lower semicontinuous. To see this, we first prove that

$$\Sigma \cap S \cap \mathrm{int}_V \Gamma(x) \neq \emptyset \quad \forall x \in X_S. \tag{7}$$

To prove (7), fix $x \in X_S$. Choose any $x_1 \in X \cap \Sigma \cap S$ such that $\|x - x_1\|_E \leq \xi/4$. Hence,

$$x - x_1 \in V_0 \cap \overline{B}_{\frac{\xi}{4}}.$$

Since by (6), we have

$$x_1 \in \overline{\mathrm{co}}\, K_2 + \left[\overline{B}_{\frac{\xi}{2}} \cap V_0\right]$$

by (5) it follows that

$$x \in \overline{\mathrm{co}}\, K_2 + \left[\overline{B}_{\frac{3\xi}{4}} \cap V_0\right] \subseteq \Sigma_1.$$

Consequently, there exists $i \in \{1, \ldots, m\}$ such that $x \in W_{z_i}$. By (3), we get in particular that $u_{z_i} \in \mathrm{int}_V \Gamma(x)$, hence

$$u_{z_i} \in S \cap \mathrm{int}_V \Gamma(x) \neq \emptyset.$$

By assumption (10), we have $\Gamma(x) \cap K_1 \neq \emptyset$. Fix any point $v \in \Gamma(x) \cap K_1$. The convexity of $\Gamma(x)$ implies that

$$v + t(u_{z_i} - v) \in S \cap \mathrm{int}_V \Gamma(x) \quad \text{for all} \quad t \in ]0, 1]. \tag{8}$$

On the other hand, since by (6), we have

$$v + \left[\overline{B}_{\frac{\xi}{2}} \cap V_0\right] \subseteq \Sigma$$

then, we can find $\alpha \in ]0,1]$ such that

$$v + t(u_{z_i} - v) \in \Sigma \quad \text{for all} \quad t \in ]0, \alpha[\,. \tag{9}$$

In particular, by (8) and (9) we have

$$S \cap \Sigma \cap \mathrm{int}_V \Gamma(x) \neq \emptyset$$

as desired. Thus, (7) is now proved. At this point, we can prove that $\Gamma_S$ is lower semicontinuous. To this aim, let $x^* \in X_S$ and let $A$ be an open set in $V$ such that

$$\Gamma_S(x^*) \cap A \neq \emptyset.$$

By (7), we have that

$$\Sigma \cap S \cap \mathrm{int}_V \Gamma(x^*) \neq \emptyset.$$

Consequently, there exists a point

$$w \in \Sigma \cap S \cap \mathrm{int}_V \Gamma(x^*) \subseteq \Gamma_S(x^*).$$

Choose a point $v^* \in A \cap \Gamma_S(x^*)$. Since, the set $\Gamma(x^*)$ is convex, we have that

$$v^* + l(w - v^*) \in X_S \cap \mathrm{int}_V \Gamma(x^*) \quad \text{for all} \quad l \in ]0,1]. \tag{10}$$

On the other hand, since $A$ is open in $V$, there exists $\mu > 0$ such that

$$v^* + \left[B_\mu \cap V_0\right] \subseteq A. \tag{11}$$

Consequently, by (10) and (11), there exists $\tau \in ]0,1]$ such that

$$v^* + \tau(w - v^*) \in X_S \cap A \cap \mathrm{int}_V \Gamma(x^*). \tag{12}$$

By Proposition 2.5 of Cubiotti [13], there is a neighborhood $Z$ of $x^*$ in $X$ such that

$$v^* + \tau(w - v^*) \in \mathrm{int}_V \left( \bigcap_{x \in Z} \Gamma(x) \right). \tag{13}$$

By (12) and (13), we get

$$v^* + \tau(w - v^*) \in X_S \cap A \cap \mathrm{int}_V \Gamma(x) \quad \text{for all} \quad x \in Z$$

hence, in particular,

$$\Gamma_S(x) \cap A \neq \emptyset \quad \text{for all} \quad x \in Z \cap X_S,$$

as desired.

(d) One has $\mathrm{aff}(\Gamma_S(x)) = \mathrm{aff}(X_S)$ for all $x \in X_S$. To see this, fix $x \in X_S$. Observe that the set

$$T := \mathrm{int}_V \Gamma(x) \cap \mathrm{aff}(X_S)$$

is open in $\mathrm{aff}(X_S)$ and by (7) one has

$$\begin{aligned}\emptyset &\neq \Sigma \cap S \cap \mathrm{int}_V \Gamma(x)\\ &= X \cap \Sigma \cap S \cap \mathrm{int}_V \Gamma(x)\\ &\subseteq X_S \cap \mathrm{int}_V \Gamma(x)\\ &\subseteq T \cap X_S,\end{aligned}$$

hence $T \cap X_S \neq \emptyset$. Consequently, by Proposition 2.1 of Cubiotti [11] (setted in the affine manifold $\mathrm{aff}(X_S)$ by an obvious translation), we get

$$\mathrm{aff}(T \cap X_S) = \mathrm{aff}(X_S). \tag{14}$$

Consequently, since

$$T \cap X_S \subseteq \Gamma_S(x) \subseteq X_S$$

by (14) we get our claim.

(e) The set $K := K_2 \cap X_S$ is compact and

$$\Gamma_S(x) \cap K \neq \emptyset \quad \text{for all} \quad x \in X_S.$$

Taking into account that $K_1 \subseteq X_S \cap K_2$, this follows easily by assumption (10) and the definition of $\Gamma_S$.

(f) For each fixed $x \in E_S \setminus K$, and each $z \in \Phi(x)$, there exists $y \in \Gamma_S(x) \cap K$ such that $\psi(x, z, y) > 0$. This follows easily by assumption (11), taking into account that $E_S \setminus K \subseteq E \setminus K_2$ and $\Gamma(x) \cap K_1 \subseteq \Gamma_S(x) \cap K$.

It is routine matter to check that all the remaining assumptions of Corollary 3.2 are satisfied. Consequently, there exists $(x_S, z_S) \in X_S \times C$ such that

$$x_S \in \Gamma_S(x_S), \quad z_S \in \Phi(x_S) \quad \text{and} \quad \psi(x_S, z_S, y) \leq 0 \quad \forall\, y \in \Gamma_S(x_S). \tag{15}$$

By (15) and assumption (11), taking into account that $K_1 \subseteq X_S$, we have that $x_S \in K_2$. We now prove that

$$\psi(x_S, z_S, y) \leq 0 \quad \text{for all} \quad y \in \Gamma(x_S) \cap S. \tag{16}$$

Indeed, if $y \in \Gamma(x_S) \cap S$, since

$$\begin{aligned}x_S \in K_2 \subseteq \overline{\mathrm{co}}\, K_2 \subseteq X &\subseteq V,\\ y \in \Gamma(x_S) \subseteq X &\subseteq V,\\ V - V &\subseteq V_0\end{aligned}$$

and $X$ is convex, we have that

$$x_S + t(y - x_S) \in X \cap \left[\overline{\mathrm{co}}\, K_2 + \left(B_{\frac{\xi}{2}} \cap V_0\right)\right] = X \cap \Sigma$$

for a sufficiently small $t \in ]0, 1[$ . Hence, by the convexity of $\Gamma(x_S)$ and by the definition of $X_S$, we have

$$x_S + t(y - x_S) \in X \cap \mathcal{S} \cap S \cap \Gamma(x_S) \subseteq X_S \cap \Gamma(x_S) = \Gamma_S(x_S).$$

By (15) and assumption (7) and (8), we get

$$0 \geq \psi(x_S, z_S, x_S + t(y - x_S)) \geq t\, \psi(x_S, z_S, y) + (1 - t)\, \psi(x_S, z_S, x_S) = t\, \psi(x_S, z_S, y)$$

hence $\psi(x_S, z_S, y) \leq 0$, as desired.

Resuming, we have proved that for each $S \in \mathcal{S}$ there exists a pair $(x_S, z_S) \in (K_2 \cap S) \times C$ such that

$$x_S \in \Gamma(x_S), \quad z_S \in \Phi(x_S) \quad \text{and} \quad \psi(x_S, z_S, y) \leq 0 \quad \forall y \in \Gamma(x_S) \cap S. \tag{17}$$

Now, we consider the net $\{x_S\}_{S \in \mathcal{S}}$, with $\mathcal{S}$ ordered by the ordinary set inclusion $\subseteq$. The compactness of $K_2$ implies that the net $\{x_S\}_{S \in \mathcal{S}}$ has a cluster point $\hat{x} \in K_2$. Since, by assumption (2) the set $E \cap K_2$ is closed, by (17), we get $\hat{x} \in \Gamma(\hat{x})$. We now claim that

$$\inf_{z \in \Phi(\hat{x})} \psi(\hat{x}, z, y) \leq 0 \quad \text{for all} \quad y \in \mathrm{int}_V \Gamma(\hat{x}). \tag{18}$$

On the contrary, assume that there exists $\tilde{y} \in \mathrm{int}_V \Gamma(\hat{x})$ such that

$$\inf_{z \in \Phi(\hat{x})} \psi(\hat{x}, z, \tilde{y}) > 0. \tag{19}$$

By Proposition 2.5 of Cubiotti [13], there exists $\sigma > 0$ such that

$$\tilde{y} \in \mathrm{int}_V \Big( \bigcap_{x \in B(\hat{x}, \sigma) \cap X} \Gamma(x) \Big). \tag{20}$$

By (19) and assumption (5), since the set

$$\Big\{ x \in E \cap K_2 : \inf_{z \in \Phi(x)} \psi(\hat{x}, z, \tilde{y}) > 0 \Big\}$$

is open in $E \cap K_2$, there exists $\alpha \in ]0, \sigma[$ such that

$$\inf_{z \in \Phi(x)} \psi(x, z, \tilde{y}) > 0 \quad \forall x \in B(\hat{x}, \alpha) \cap K_2 \cap E. \tag{21}$$

By construction, there exists $\hat{S} \in \mathcal{S}$ such that $\tilde{y} \in \hat{S}$ and $x_{\hat{S}} \in B(\hat{x}, \alpha)$. By (20) we get $\tilde{y} \in \Gamma(x_{\hat{S}}) \cap \hat{S}$. Consequently, (17) implies that

$$\psi(x_{\hat{S}}, z_{\hat{S}}, \tilde{y}) \leq 0. \tag{22}$$

On the other hand, (21) implies that

$$\inf_{z \in \Phi(x_{\hat{S}})} \psi(x_{\hat{S}}, z, \tilde{y}) > 0$$

hence, in particular,

$$\psi(x_{\hat{S}}, z_{\hat{S}}, \tilde{y}) > 0,$$

which contradicts (22). Consequently, (18) holds, hence

$$\sup_{y \in \mathrm{int}_V \Gamma(\hat{x})} \inf_{z \in \Phi(\hat{x})} \psi(\hat{x}, z, y) \leq 0.$$

By Theorem 5 at p216 of Aubin [1], taking into account assumptions (1), (4), (8) and (9), we then get

$$\inf_{z \in \Phi(\hat{x})} \sup_{y \in \mathrm{int}_V \Gamma(\hat{x})} \psi(\hat{x}, z, y) \leq 0. \tag{23}$$

Since by assumption (10) the function

$$z \to \sup_{y \in \mathrm{int}_V \Gamma(\hat{x})} \psi(\hat{x}, z, y)$$

is lower semicontinuous on $\Phi(\hat{x})$ and the last set is compact, by (23) we get the existence of a point $\hat{z} \in \Phi(\hat{x})$ such that

$$\sup_{y \in \mathrm{int}_V \Gamma(\hat{x})} \psi(\hat{x}, \hat{z}, y) \leq 0.$$

Taking into account assumption (viii), this implies

$$\sup_{y \in \Gamma(\hat{x})} \psi(\hat{x}, \hat{z}, y) \leq 0.$$

The proof is now complete. □

**Example** Let $E = C = \mathbb{R}^2$, $X = [0,1] \times [0,1]$, $\Gamma$, $\Phi$ and $\psi$ be defined as follows: $\Gamma(x) = X$, $\Phi(x) = \{(1,1)\}$ if $x = (0,0)$, $\Phi(x) = [0, 1/\|x\|^2] \times \{1\}$ if $x \neq (0,0)$ and $\psi(x,z,y) = \langle z, x-y \rangle$. Then all assumptions in Theorem 3.3 are satisfied and it can be easily seen that $(\hat{x}, \hat{z}) \in X \times C$ is a solution of (1) where $\hat{x} = (0,0)$ and $\hat{z} = (1,1)$. But, we observe that $\Phi$ is neither upper semicontinuous nor lower semicontinuous on $X$.

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